# The theory of cosmic ray acceleration and transport III. Test particle transport

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**Topics**:

1. Fundamental questions
2. General particle transport equations
3. Fokker-Planck transport equation
4. Fokker-Planck coefficients
5. Diffusion approximation
6. Focused acceleration
7. Momentum spectra
8. Summary and conclusions

**Literature:**

Diffusion, Scattering, and Acceleration of Particles by Stochastic Electromagnetic Fields, D. E. Hall, P. A.Sturrock, 1967, Phys. Fluids 10, 2620

A new cosmic ray transport theory in partially turbulent space plasmas: Extending the quasilinear approach; RS, 2011, ApJ 732, 96

# 1. Fundamental questions

Which equations describe the dynamics of cosmic rays for given and specified electromagnetic fields ($test$-$particle\ approach$)?

How do the cosmic ray transport parameters depend on the statistical properties of the turbulent electromagnetic fields in space?

Under which conditions is cosmic ray transport diffusive?

What causes parallel and perpendicular spatial diffusion and the acceleration of CRs?

Numerical cosmic ray transport codes such as DRAGON, CR-PROPA, GALPROP and PICARD numerically solve the diffusion-convection transport equation containing diffusion and convection terms in the particles$'$ momentum and space coordinates. Are all important physical effects represented?

## 1.1. Methods used

Because of the complicated nonlinear equations of motion of charged particles in partially random electromagnetic fields there are only two methods to study theoretically particle acceleration and transport: (i) numerical PIC (particle-in-cell) simulations of highly idealized configurations, (ii) quasilinear perturbation theory valid for small turbulence levels $|\delta\vec{B}| \ll \vec{B}_0$. Both have their advantages and shortcomings, and they complement each other.

Obviously, besides limited computer power numerical simulations require the precise knowledge of many important input plasma parameters as well as the specification of initial and boundary conditions which at least for the more distant cosmic objects are not known. By chosing the wrong input plasma quantities one may end up in an irrelevant range of solution space. Of course, when all these input quantities are known and given, the simulations result in a very accurate and detailed description of the acceleration processes on all spatial, momentum and time scales of interest.

After its original developments for longitudinal plasma waves (Vedenov et al. 1962) the application of quasilinear theory to astrophysical plasmas has turned out to be very fruitful in explaining the dynamics of energetic charged particles in these plasmas. Quasilinear transport equations for magnetohydrodynamic plasma waves were pioneered by Kennel and Engelmann (1966), Jokipii (1966), Hall and Sturrock (1967), Lerche (1968) and Kulsrud and Pearce (1969).

The quasilinear approach to the interaction of charged particles with partially random electromagnetic fields is a first-order perturbation calculation in the ratio $q_L = (\delta B/B_0)^2$ and requires values of this ratio $q_L \leq 4$. In most cosmic plasmas this requirement is well satisfied as has been established by direct in-situ measurements in interplanetary plasmas, or due to saturation effects in the growth of fluctuating fields. The standard quasilinear approach also requires incoherent mode coupling of the fluctuating electromagnetic fields described as the superposition of individual plasma wave modes.

## 2. General particle transport equations

We start from the equation of motions of charged particles in a medium at rest

$$\frac{d\vec{x}}{dt} = \vec{v} = \frac{\vec{p}}{\gamma m}, \quad \gamma = \sqrt{1+\frac{p^2}{m^2c^2}},$$

$$\frac{d\vec{p}}{dt} = q_a\left[\delta\vec{E} + \frac{\vec{v}\times(\vec{B}_0+\delta\vec{B})}{c}\right] = q_a\left[\delta\vec{E} + \frac{\vec{p}\times(\vec{B}_0+\delta\vec{B})}{\gamma m_a c}\right] \tag{1}$$

We orient the large-scale guide magnetic field, which is uniform on the scales of the cosmic ray particles gyradii $R_L = v/|\Omega|$, $\vec{B}_0 = B_0\vec{e}_z = (0,0,B_0)$ along the $z$-axis. Let $\Omega_a = q_a B_0/\gamma m_a c$ denote the relativistic gyrofrequency.

Because of the gyrorotation of the particles in the uniform magnetic field, one is not so much interested in their actual position as in the coordinates of the guiding center

$$\vec{X} = (X,Y,Z) = \vec{x} + \frac{\vec{v}\times\vec{e}_z}{\Omega_a} = \vec{x} + \frac{c}{q_a B_0}\vec{p}\times\vec{e}_z = \vec{x} + \frac{c}{q_a B_0}\begin{pmatrix} p_y \\ -p_x \\ 0 \end{pmatrix} \tag{2}$$

We transform from the phase space variables $(x,y,z,p_x,p_y,p_z)$ to the guiding center spatial coordinates (2) and spherical momentum coordinates

$$\phi = \arctan(p_y/p_x), \ \mu = \arccos(p_z/p), \ \ p = (p_x^2+p_y^2+p_z^2)^{1/2} \tag{3}$$

The kinetic Klimontovich equation then reads

$$\frac{\partial f_a}{\partial t} + v\mu \frac{\partial f_a}{\partial Z} - \Omega_a \frac{\partial f_a}{\partial \phi} + p^{-2} \frac{\partial}{\partial y_\alpha} \left[ p^2 h_\alpha(t) f_a \right] - Q_0(Z, X, Y, p, \mu, \phi, t) = 0, \tag{4}$$

where we use the Einstein sum convention for indices, and $y_\alpha \in [\mu, p, \phi, X, Y]$ represent the five phase space variables with non-vanishing stochastic fields $h_\alpha(t)$ from $\delta\vec{E}$ and $\delta\vec{B}$.

$$Q_0(z, X, Y, p, \mu, \phi, t) = S_0(z, X, Y, p, \mu, \phi, t) - \mathcal{N}_0 F - \mathcal{R}_0 F \tag{5}$$

accounts for sources and sinks ($S_0$) and the effects of the mirror force ($\mathcal{N}_0$) and momentum loss processes ($\mathcal{R}_0$), where the latter two operate on much longer spatial and time scales than the particle interactions with the stochastic fields.

The equations of motion (1) provide for the time evolution of the particle momentum $p$, the pitch-angle cosine $\mu$ and phase $\phi$

$$\frac{dp}{dt} = h_p(t) = \frac{q_a}{p}\vec{p}\cdot\delta\vec{E} = q_a\left(\mu\delta E_z + \sqrt{1-\mu^2}\,[\cos\phi\delta E_x + \sin\phi\delta E_y]\right), \quad (6)$$

$$\frac{d\mu}{dt} = h_\mu(t) = \frac{q_a\delta E_z}{p} - \frac{\mu}{p}h_p(t) + \frac{\Omega_a}{B_0}\sqrt{1-\mu^2}\,(\cos\phi\delta B_y - \sin\phi\delta B_x)\,, \quad (7)$$

$$\frac{d\phi}{dt} = -\Omega_a + h_\phi(t), \quad h_\phi(t) = -\frac{\Omega_a}{B_0}\delta B_z + \frac{\Omega\mu}{B_0\sqrt{1-\mu^2}}(\cos\phi\delta B_x + \sin\phi\delta B_y)$$
$$+\frac{q_a}{p\sqrt{1-\mu^2}}(\delta E_y\cos\phi - \delta E_x\sin\phi)\,. \quad (8)$$

with the three random forces $h_p(t)$, $h_\mu(t)$ and $h_\phi(t)$. Eq. (8) also accounts for the regular force term $\dot{\phi} = -\Omega$ in Eq. (4).
Likewise, for the guiding center coordinates

$$\frac{dX}{dt} = h_X(t) = \frac{c\delta E_y}{B_0} + \frac{p}{\gamma m_a B_0}[\mu\delta B_x - \sqrt{1-\mu^2}\cos\phi\delta B_z],$$
$$\frac{dY}{dt} = h_Y(t) = -\frac{c\delta E_x}{B_0} + \frac{p}{\gamma m_a B_0}[\mu\delta B_y - \sqrt{1-\mu^2}\sin\phi\delta B_z], \quad (9)$$

$h_Z(t) = 0$, and the regular force $\dot{Z} = v\mu$ accounted for in Eq. (4).

## 2.1. Ensemble averaging and quasilinear approximation

The distribution function $f_a = f$ (in the following we drop the index $a$) in Eq. (4) develops in an irregular way under the influence of the five stochastic force fields $h_\alpha(t)$, but the detailed fluctuations are not of interest.

We seek an expectation value of $f$ in terms of the statistical properties of $h_\alpha(t)$, so we consider an ensemble of distribution functions all beginning with identical values at time $t_0$. Let each of these functions be subject to a different member of an ensemble of realizations of $h_\alpha(t)$, i.e. fluctuating field histories which are independent of one another in detail, but identical as to statistical averages. At any time $t > t_0$, the various functions differ from each other, and we require an equation for $< f >$, the average of $f$ over all members of the ensemble.

With $f =< f > +\delta N$, where $\delta N$ denotes the deviation from the ensemble average, and the regular gyrocenter force operator

$$\mathcal{L}_{0,g} = \frac{\partial}{\partial t} + v\mu \frac{\partial}{\partial Z} - \Omega \frac{\partial}{\partial \phi}, \tag{10}$$

Eq. (4) reads

$$\mathcal{L}_{0,g} < f > + \mathcal{L}_{0,g}\delta N + p^{-2}\frac{\partial}{\partial y_\alpha}\left[p^2 h_\alpha(t) < f >\right] + p^{-2}\frac{\partial}{\partial y_\alpha}\left[p^2 h_\alpha(t)\delta N\right] - Q_a = 0 \tag{11}$$

Ensemble-averaging Eq. (11) using $< h_\alpha(t) >=< \delta N >= 0$ then yields

$$\mathcal{L}_{0,g} < f > -Q_a = -p^{-2} \frac{\partial}{\partial y_\alpha} \left[ p^2 < h_\alpha(t) \delta N > \right] , \tag{12}$$

Substracting Eq. (12) from Eq. (11) gives the equation for the deviation (see lecture 2)

$$\mathcal{L}_{0,g}(\delta N - N^0) = -h_\alpha(t) \frac{\partial}{\partial y_\alpha} < f > -h_\alpha(t) \frac{\partial}{\partial y_\alpha} \delta N + < h_\alpha(t) \frac{\partial}{\partial y_\alpha} \delta N >, \tag{13}$$

where we used the property $p^{-2} \frac{\partial}{\partial y_\alpha} (p^2 h_\alpha(t)) = 0$.
With the inverted regular time-integration operator $\mathcal{L}_{0,g}^{-1}$ the formal solution of Eq. (13) is given by

$$\delta N - N^0 = -\mathcal{L}_{0,g}^{-1} h_\sigma(t) \frac{\partial < f >}{\partial y_\sigma} - \mathcal{L}_{0,g}^{-1} h_\sigma(t) \frac{\partial \delta N}{\partial y_\sigma} + \mathcal{L}_{0,g}^{-1} < h_\sigma(t) \frac{\partial \delta N}{\partial y_\sigma} >, \tag{14}$$

where we changed the summation index. The last term in Eq. (14) can be ignored as it does not contribute to the ensemble average on the right hand side of Eq. (12). Eq. (14) has the series solution

$$\delta N - N^0 = -\mathcal{L}_{0,g}^{-1} h_\sigma(t) \frac{\partial < f >}{\partial y_\sigma} - \mathcal{L}_{0,g}^{-1} h_\sigma(t) \frac{\partial}{\partial y_\sigma} [N^0 - \mathcal{L}_{0,g}^{-1} h_\eta(t) \frac{\partial < f >}{\partial y_\eta}] + \dots \tag{15}$$

Within the quasilinear approximation we only keep terms of first order in stochastic fields, so that


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$$\delta N - N^0 \simeq -\mathcal{L}_{0,g}^{-1} h_\sigma(t) \frac{\partial < f >}{\partial y_\sigma} \tag{16}$$

## 2.2. Inverted regular force operator

The inverted regular force operator $\mathcal{L}_{0,g}^{-1}$ is obtained by integrating along the characteristics of the operator $\mathcal{L}_{0,g}$ (Achatz et al. 1991), which is the unperturbed gyrocenter orbit $(X_u = X_0,\ Y_u = Y_0,\ Z_u = Z_0 + v\mu(t-t_0),\ p_u = p_0,\ \mu_u = \mu_0,\ \phi_u = \phi_0 - \Omega(t-t_0))$. We consider

$$\mathcal{L}_{0,g} \delta A(\vec{X}, \vec{p}, t) = -h_\alpha(\vec{x}, \vec{p}, t) \frac{\partial}{\partial y_\alpha} < f > (t), \quad \delta A = \delta N - N^0 \tag{17}$$

and introduce the Fourier transforms in space[1]

$$\delta A(\vec{X}, \vec{p}, t) = \int d^3k A_1(\vec{k}, \vec{p}, t) e^{\imath \vec{k}\cdot\vec{X}}, \ \delta h_\alpha(\vec{x}, \vec{p}, t) = \int d^3k H_\alpha(\vec{k}, \vec{p}, t) e^{\imath \vec{k}\cdot\vec{x}}$$
$$= \int d^3k H_\alpha(\vec{k}, \vec{p}, t) \exp\left[\imath \vec{k}\cdot\vec{X} + \frac{\imath}{\Omega}(k_y v_x - k_x v_y)\right]$$
$$= \int d^3k H_\alpha(\vec{k}, \vec{p}, t) \exp\left[\imath \vec{k}\cdot\vec{X} + \frac{\imath k_\perp v \sqrt{1-\mu^2} \sin(\psi - \phi)}{\Omega}\right], \tag{18}$$

[1] It is important to distinguish between the gyrocenter ($\vec{X}$) and and particle ($\vec{x}$) coordinates.

where we use Eq. (2) and introduce cylindrical coordinates for the wave vector

$$\vec{k} = (k_\perp \cos\psi, k_\perp \sin\psi, k_\parallel) \tag{19}$$

After a little algebra we obtain

$$\delta N(\vec{X},\vec{p},t) = N^0 - \mathcal{T} h_\sigma(s)\frac{\partial < f > (s)}{\partial y_\sigma} = N^0 - \int d^3k e^{\imath\vec{k}\cdot\vec{X}} \int_{t_0}^{t} ds\, H_\sigma(\vec{k},\vec{p},s)$$

$$\times\, e^{\imath k_\parallel v\mu(s-t) + \frac{\imath k_\perp v\sqrt{1-\mu^2}\sin(\psi-\phi_0+\Omega(s-t_0))}{\Omega}} \frac{\partial < f > (s)}{\partial y_\sigma} \tag{20}$$

Then the ensemble average on the right hand side of Eq. (12) can be readily calculated as

$$< h_\alpha(t)\delta N > = < h_\alpha(t)N^0 > - < h_\alpha(t)\mathcal{T}h_\sigma(s) > \frac{\partial < f > (s)}{\partial y_\sigma}, \tag{21}$$

yielding the quasilinear transport equation for the ensemble-averaged $< f >$:


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$$\mathcal{L}_{0,g} < f >= -p^{-2} \frac{\partial}{\partial y_\alpha} \left[ p^2 < h_\alpha(t) N^0 > \right]$$
$$+ p^{-2} \frac{\partial}{\partial y_\alpha} \left[ p^2 < h_\alpha(t) \mathcal{T} h^*_\sigma(s) > \frac{\partial < f > (s)}{\partial y_\sigma} \right], \tag{22}$$

where we have replaced $h_\sigma(s) = h^*_\sigma(s)$ by its complex conjugate, because the stochastic forces are real-valued quantities. The first term represents the quasilinear drag term ($\propto < f >$) discussed in lecture 2. The 2nd term involves a complicated integro-differential operator. In the following we ignore the drag term.

In order to get from Eq. (22) an useful differential equation for $< f >$, we employ a number of approximations. the **adiabatic approximation** and the assumptions of **homogeneous and quasi-stationary turbulence**. This leads us to the Fokker-Planck transport equation for CRs, which is diffusive, if a **small enough finite decorrelation time of second-order correlation functions exists**.

# 3. Fokker-Planck transport equation

## 3.1. Step 1: Adiabatic approximation

We follow the adiabatic approximation of Hall and Sturrock (1967) and Achatz et al. (1991), that $< f > (s) \simeq < f > (t)$ varies only negligibly over the time $s$-integration interval, providing for Eq. (22) the Fokker-Planck equation

$$\mathcal{L}_0 < f > (t) - Q_a = p^{-2} \frac{\partial}{\partial y_\alpha} \left[ p^2 \bar{P}_{\alpha\sigma} \frac{\partial < f > (t)}{\partial y_\sigma} \right], \qquad (23)$$

with the full Fokker-Planck coefficients $\bar{P}_{\alpha\sigma} =< h_\alpha(t) \mathcal{T} h^*_\sigma(s) >$. Using again the Fourier transform (18) in the form

$$\delta h_\alpha(\vec{x}, \vec{p}, t) = \int d^3k_1 \, H_\alpha(\vec{k}_1, \vec{p}, t) \exp \left[ \imath \vec{k}_1 \cdot \vec{X} + \frac{\imath k_{1,\perp} v \sqrt{1-\mu^2} \sin(\psi_1 - \phi(t))}{\Omega} \right], \qquad (24)$$

we obtain with Eq. (20)

$$\bar{P}_{\alpha\sigma} = \int d^3k_1 \int d^3k \, e^{\imath(\vec{k}_1 - \vec{k}) \cdot \vec{X} + \frac{\imath k_{1,\perp} v \sqrt{1-\mu^2} \sin(\psi_1 - \phi_0 + \Omega(t-t_0))}{\Omega}} \int_{t_0}^{t} ds$$

$$\times < H_\alpha(\vec{k}_1, \vec{p}, t) H_\sigma(\vec{k}, \vec{p}, s) e^{-\imath k_\parallel v \mu (s-t) - \frac{\imath k_\perp v \sqrt{1-\mu^2} \sin(\psi - \phi_0 + \Omega(s-t_0))}{\Omega}} >, \qquad (25)$$

where we use that at time $s = t$ the CR particle phase is given by $\phi(t) = \phi_0 + \Omega(t - t_0)$. Eq. (25) involves 7 integrals.

## 3.2. Step 2: Homogeneous turbulence

As second assumption we use that *the turbulent electric and magnetic fields are homogenously distributed*, meaning that independent from the position of the gyrocenter $\vec{X}$ the particles are subject to turbulence realizations with the same statistical properties. This allows us to average the full Fokker-Planck coefficients (25) over the spatial position of the guiding center using

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3X \; e^{\imath(\vec{k}_1-\vec{k})\cdot\vec{X}} = \delta(\vec{k}_1-\vec{k}), \tag{26}$$

implying that turbulence fields at different wavevectors are uncorrelated. We obtain for Eq. (25) (only 4 intergrals left)

$$P_{\alpha\sigma} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3X \; \bar{P}_{\alpha\sigma} = \int d^3k \, e^{\frac{\imath k_\perp v\sqrt{1-\mu^2}\sin(\psi-\phi_0+\Omega(t-t_0))}{\Omega}} \int_{t_0}^{t} ds$$
$$\times < H_\alpha(\vec{k},\vec{p},t) H_\sigma(\vec{k},\vec{p},s) e^{-\imath k_\parallel v\mu(s-t)-\frac{\imath k_\perp v\sqrt{1-\mu^2}\sin(\psi-\phi_0+\Omega(s-t_0))}{\Omega}} > \tag{27}$$

## 3.3. Step 3: Quasi-stationary turbulence

Here the correlation functions $< H_\alpha(t)H^*_\sigma(s) >$ in Eq. (27) depend only on the absolute value of the time difference $|\tau| = |t-s|$, so that with the substitution $s = t-\tau$ we have $< H_\alpha(t)H^*_\sigma(t-\tau) >=< H_\alpha(0)H^*_\sigma(-\tau) >$, implying for the Fokker-Planck coefficients (27)

$$P_{\alpha\sigma} = \int d^3k \int_0^{t-t_0} d\tau \ < H_\alpha(\vec{k},0) H^*_\sigma(\vec{k},-\tau) \exp\left[\imath v\mu k_\parallel \tau\right]$$

$$\times \exp\left[\frac{\imath k_\perp v\sqrt{1-\mu^2}}{\Omega}\left(\sin(\phi_1-\psi+\Omega\tau)-\sin(\phi_1-\psi)\right)\right] >, \qquad (28)$$

where $\phi_1 = \phi_0 + \Omega t_0$ includes the irrelevant constant $\Omega t_0$.

## 3.4. Step 4: Finite turbulence decorrelation time

If a *finite decorrelation time* $t_c$ exists, such that the correlation functions $< H_\alpha(0)H^*_\sigma(-\tau) > \to 0$ fall to a negligible magnitude for $\tau \to \infty$, we are allowed to replace the upper integration boundary in the $\tau$-integral in Eq. (28) by infinity so that

$$P_{\alpha\sigma} = \int_0^\infty d\tau \ < h_\alpha(0) h^*_\sigma(-\tau) >= \int d^3k \int_0^\infty d\tau \ < H_\alpha(\vec{k},0) H^*_\sigma(\vec{k},-\tau)$$

$$\times \exp\left[\imath v\mu k_\parallel \tau + \frac{\imath k_\perp v\sqrt{1-\mu^2}}{\Omega}\left(\sin(\phi_1-\psi)-\sin(\phi_1-\psi+\Omega\tau)\right)\right] > \quad (29)$$

## 3.5. Step 5: Ensemble averaging by initial random phases

The brackets $< \ldots >$ in Eq. (29) indicate that the Fokker-Planck coefficients have to be ensemble-averaged over different realizations of the stochastic fields. The appropriate averaging variable is the phase $\phi_1$, determined apart from a constant by the initial phase of the CR particles, which is a random variable that can take on any value between $0$ and $2\pi$. We identify for any quantity $A(\phi_1)$ that $< A(\phi_1) >= (2\pi)^{-1} \int_0^{2\pi} d\phi_1 \, A(\phi_1)$. The Fokker-Planck coefficients (29) then read

$$P_{\alpha\sigma} = \frac{1}{2\pi} \int d^3k \int_0^{\infty} d\tau \int_0^{2\pi} d\phi_1 \, C_{\alpha,\sigma}(\vec{k}, \tau)$$
$$\times \exp\left[ \imath v \mu k_{\parallel} \tau + \frac{\imath k_{\perp} v \sqrt{1-\mu^2}}{\Omega} \left( \sin(\phi_1 - \psi) - \sin(\phi_1 - \psi + \Omega\tau) \right) \right] \quad (30)$$

with the respective second-order correlation functions of the stochastic forces $C_{\alpha,\sigma}(\vec{k}, \tau) = H_{\alpha}(\vec{k}, 0) H^*_{\sigma}(\vec{k}, -\tau)$.

The Fokker-Planck coefficients (30) are of so-called Taylor-Green-Kubo (TGK) form (Taylor 1922, Green 1951, Kubo 1957). As demonstrated, the two assumptions of quasi-stationary homogeneous turbulence and the existence of a finite turbulence decorrelation time $t_c$ guarantee diffusive transport behaviour. The Fokker-Planck transport equation (23) becomes

$$\mathcal{L}_{0,g} < f > (t) - Q_a = p^{-2} \frac{\partial}{\partial y_{\alpha}} \left[ p^2 P_{\alpha\sigma} \frac{\partial < f > (t)}{\partial y_{\sigma}} \right] \quad (31)$$

## 3.6. Fourier transforms of the stochastic fields

The individual Fourier transforms of the five stochastic fields $h_\alpha(t)$ can be calculated from Eq. (18), yielding e.g.

$$H_p(\vec{k},0) = \frac{\Omega pc}{vB_0}\Big(\mu\delta E_z(\vec{k},0) + \sqrt{1-\mu^2}\big[\cos(\phi_1-\psi)\delta E_x(\vec{k},0) + \sin(\phi_1-\psi)\delta E_y(\vec{k},0)\big]\Big),$$

$$H_p^*(\vec{k},-\tau) = \frac{\Omega pc}{vB_0}q_a\Big(\mu\delta E_z^*(\vec{k},-\tau) + \sqrt{1-\mu^2}\big[\cos(\phi_1-\psi+\Omega\tau)\delta E_x^*(\vec{k},-\tau) + \sin(\phi_1-\psi+\Omega\tau)\delta E_y^*(\vec{k},-\tau)\big]\Big), \tag{32}$$

In general, 25 different correlation functions $C_{\alpha,\sigma}(\tau)$ and Fokker-Planck coefficients $P_{\alpha,\sigma}$ result involving different magnetic and electric correlation functions. Depending on the properties of the chosen electromagnetic turbulence model, not all of these 25 Fokker-Planck coefficients are nonzero, and some of them are much larger than others.

The induction law relates

$$\vec{k}\times\delta\vec{E}(\vec{k},t) = \frac{\imath}{c}\partial_t\delta\vec{B}(\vec{k},t) \tag{33}$$

As shown in lecture 2 magnetized space plasmas contain low-frequency linear ($\delta B \ll B_0$) transverse MHD waves (such as shear Alfven and magnetosonic plasma waves) with dispersion relations $\omega_R^2 = V_A^2 k_\parallel^2$ and $\omega_R^2 = V_A^2 k^2$, respectively. The induction law (33) then indicates for MHD waves $\delta E = (V_A/c)\delta B$.

We estimate the relative strength of the stochastic forces (32), adopting values of order unity for $\mu$, $\sqrt{1-\mu^2}$, $\cos(\phi_1 - \psi + \Omega\tau)$ and $\sin(\phi_1 - \psi + \Omega\tau)$. For energetic CR particles with $v \gg V_A$

$$H_p \simeq \Omega \frac{\delta B}{B_0} p(V_A/v), \;\; H_\mu \simeq H_\phi \simeq \Omega \frac{\delta B}{B_0}, \;\; H_{X,Y} \simeq v \frac{\delta B}{B_0} \tag{34}$$

The corresponding Fokker-Planck coefficients then scale as

$$D_{\mu\mu} \simeq D_{\phi\phi} \simeq D_0 = \Omega^2 \frac{\delta B^2}{B_0^2}, \; D_{pp} \simeq D_0 \frac{V_A^2 p^2}{v^2}, \; D_{X,Y} \simeq R_L^2 D_0,$$

$$D_{\mu p} \simeq D_{\phi p} \simeq D_0 \frac{V_A p}{v}, \; D_{\mu X} \simeq D_{\phi X} = R_L D_0 \tag{35}$$

Consequently, the associated times scales for pitch-angle scattering ($T_\mu \simeq D_{\mu\mu}^{-1}$), gyrophase scattering ($T_\phi \simeq D_{\phi\phi}^{-1}$), momentum diffusion ($T_p \simeq p^2/D_{pp}$) and perpendicular spatial gyrocenter diffusion ($T_X \simeq X^2/D_{XX}$) scale as

$$T_\mu \simeq T_\phi = T_0 \simeq D_0^{-1}, \;\; T_p \simeq \frac{v^2}{V_A^2} T_0 \gg T_0, \;\; T_X \simeq \frac{X^2}{R_L^2} T_0 \gg T_0 \tag{36}$$

Therefore, in the presence of low-frequency MHD fluctuations the particles will relax most quickly on the time scale $\min[\Omega^{-1}, T_0]$ to an isotropic, gyrotropic distribution function, which then on considerably longer time scales $T_X$ and $T_p$ undergoes diffusion in position space and momentum space, respectively.

Hence, a perturbation scheme based on $B_0 \gg \delta B \gg \delta E$ corresponds to the reduction

$$< f > (\vec{X}, p, \mu, \phi, t) \to f_0(\vec{X}, p, \mu, t) \to F(\vec{X}, p, t) \tag{37}$$

to gyrotropic $f_0(\vec{X}, p, \mu, t)$ and to isotropic, gyrotropic distributions functions $F(\vec{X}, p, t)$, respectively, in excellent agreement with the observed isotropy of CRs.

## 3.7. Strongly magnetized systems

For strongly magnerized systems $\delta B \ll B_0$ we employ the *small Larmor radius approximation* (Chew et al. 1956, Kennel and Engelmann 1962) that all changes are considered small over space scales comparable with the particle Larmor radii or time scales comparable with typical gyroperiods. Therefore the Larmor radius and gyroperiod are convenient small expansion parameters. The Larmor orbiting of particles is so rapid that all inhomogeneities in the $\phi$-distribution of particles are smoothed out on the macroscopic scale, and the distribution functions are independent of the gyrophase $\phi$ to lowest order.

Singleing out the phase space variable $\phi$ by introducing the reduced set of variables $x_{\alpha,\sigma} \in [X, Y, \mu, p]$ and using the explicit form of the regular force operator (10), the Fokker-Planck transport equation (31) is

$$\partial_t < f > +v\mu\partial_Z < f > -\Omega\partial_\phi < f > -Q_a = p^{-2}\frac{\partial}{\partial x_\alpha}p^2 P_{\alpha\sigma}\frac{\partial < f >}{\partial x_\sigma}$$
$$+\frac{\partial}{\partial\phi}P_{\phi\sigma}\frac{\partial < f >}{\partial x_\sigma} + p^{-2}\frac{\partial}{\partial x_\alpha}p^2 P_{\alpha\phi}\frac{\partial < f >}{\partial\phi} + \frac{\partial}{\partial\phi}P_{\phi\phi}\frac{\partial < f >}{\partial\phi} \quad (38)$$

With the expansion $< f >= f_0 + \Omega^{-1} f_1$ inserted in Eq. (38) we then find to lowest order $\partial f_0/\partial\phi = 0$. Thus the lowest-order distribution function $f_0$ is independent of the gyrophase $\phi$. To find the spatial and time dependence of $f_0$ we go to next order giving

$$\partial_t f_0 + v\mu\partial_Z f_0 - \partial_\phi\left[f_1 - P_{\phi\sigma}\frac{\partial f_0}{\partial x_\sigma}\right] - Q_a = p^{-2}\frac{\partial}{\partial x_\nu}p^2 P_{\nu\sigma}\frac{\partial f_0}{\partial x_\sigma} \quad (39)$$

The physical requirement that $f_1$ be periodic in $\phi$ then removes the third term on the left hand side when averaging Eq. (39) from $0$ to $2\pi$ in $\phi$, leading to the Larmor-phase-averaged quasilinear Fokker-Planck transport equation

$$\partial_t f_0 + v\mu\partial_z f_0 - Q(X, Y, Z, p, \mu, t) = p^{-2}\frac{\partial}{\partial x_\alpha}p^2 D_{\alpha\sigma}\frac{\partial f_0}{\partial x_\sigma} \quad (40)$$

with the gyro-averaged source term

$$Q(Z,X,Y,p,\mu,t) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, Q_a(z,X,Y,p,\mu,\phi,t), \qquad (41)$$

and the gyro-averaged Fokker-Planck coefficients

$$D_{\alpha\sigma} = \frac{1}{2\pi}\int_0^{2\pi} d\phi P_{\alpha\sigma} = \frac{1}{4\pi^2}\int d^3k \int_0^{\infty} d\tau \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi_1\, C_{\alpha,\sigma}(\vec{k},\tau)$$
$$\times \exp\left[\imath v\mu k_{\parallel}\tau + \frac{\imath k_{\perp} v\sqrt{1-\mu^2}}{\Omega}\left(\sin(\phi_1-\psi) - \sin(\phi_1-\psi+\Omega\tau)\right)\right] \qquad (42)$$

Inserting the source term (5) in Eq. (40) provides for the Larmor-phase-averaged Fokker-Planck transport equation (Schlickeiser and Jenko 2010)

$$\frac{\partial f_0}{\partial t} + v\mu\frac{\partial f_0}{\partial Z} + \mathcal{N} f_0 + \mathcal{R} f_0 - S(\vec{X},p,\mu,t) = p^{-2}\frac{\partial}{\partial x_\alpha} p^2 D_{\alpha\sigma}\frac{\partial f_0}{\partial x_\sigma}, \qquad (43)$$

where (with charge sign $\epsilon_a = q_a/|q_a|$)

$$\mathcal{N} f_0 = \frac{v(1-\mu^2)}{2} \left[ \frac{1}{L_3} \frac{\partial f_0}{\partial \mu} + \epsilon_a R_L \left( \frac{1}{L_2} \frac{\partial f_0}{\partial X} - \frac{1}{L_1} \frac{\partial f_0}{\partial Y} \right) \right] \tag{44}$$

accounts for the effects of the mirror force in the large spatial gradients ($L_1^{-1} = -\partial_x \ln B_0$, $L_2^{-1} = -\partial_y \ln B_0$, $L_3^{-1} = -\partial_z \ln B_0$) of the guide field, and

$$\mathcal{R} f_0 = p^{-2} \partial_p \left[ p^2 \dot{p}_{\rm loss} f_0 \right] + \frac{f_0}{T_c} \tag{45}$$

representing continuous ($\dot{p}_{\rm loss}$) and catastrophic ($T_c$) momentum losses of particles. $S(\vec{X}, p, \mu, t)$ represents additional sources and sinks of particles.

## 3.8. Relativistic flows

The Fokker-Planck equation (43) holds in the comoving frame of reference, i.e. the rest system of the moving plasma supporting the electromagnetic fluctuations. In astrophysics we deal very often with outflow sources, where the background plasma, supporting the plasma fluctuations, moves with respect to the observer with the bulk speed $\vec{U} = U\vec{e}_z$ along the ordered magnetic field $\vec{B}_0 = B_0 \vec{e}_z$.

In such cases, it is convenient to transform to the mixed comoving coordinate system (Kirk et al. 1988), in which time $t^*$ and the space coordiantes $\vec{x}^*$ are measured in the laboratory (=observer) system and the particle's momentum coordinates $\vec{p}$ are measured in the rest frame of the streaming plasma. This is particularly important for relativistic flows such as gamma-ray burst sources and the jets of active galactic nuclei. Consequently, the kinetic equation (43) must be transformed into these variables providing (Webb 1985, Kirk et al. 1988)

$$\Gamma\left[1+\frac{U^* v\mu}{c^2}\right]\left[\frac{\partial f_0}{\partial t^*}-\frac{1}{c^2}\frac{\partial U^*}{\partial t^*}\Gamma^2 E\frac{\partial f_0}{\partial p_z}\right]+\Gamma\left[U^*+v\mu\right]\left[\frac{\partial f_0}{\partial z^*}-\frac{1}{c^2}\frac{\partial U^*}{\partial z^*}\Gamma^2 E\frac{\partial f_0}{\partial p_z}\right]$$

$$+\mathcal{N}f_0+\mathcal{R}f_0-S(\vec{x}^*,p,\mu,t^*)=\frac{1}{p^2}\frac{\partial}{\partial x_\alpha}p^2 D_{\alpha\sigma}\frac{\partial f_0}{\partial x_\sigma}, \qquad (46)$$

where $E=pc^2/v$ denotes the particle energy and $\Gamma=(1-(U^{*2}/c^2))^{-1/2}$ and

$$\frac{\partial}{\partial p_z}=\mu\frac{\partial}{\partial p}+\frac{1-\mu^2}{p}\frac{\partial}{\partial \mu} \qquad (47)$$

Consequently, Eq. (46) becomes

$$\Gamma\left[1+\frac{Uv\mu}{c^2}\right]\left[\frac{\partial f_0}{\partial t}-\frac{1}{v}\frac{\partial U}{\partial t}\Gamma^2\left(\mu p\frac{\partial f_0}{\partial p}+(1-\mu^2)\frac{\partial f_0}{\partial \mu}\right)\right]$$
$$+\Gamma\left[U+v\mu\right]\left[\frac{\partial f_0}{\partial Z}-\frac{1}{v}\frac{\partial U}{\partial z}\Gamma^2\left(\mu p\frac{\partial f_0}{\partial p}+(1-\mu^2)\frac{\partial f_0}{\partial \mu}\right)\right]$$
$$+\mathcal{N}f_0+\mathcal{R}f_0-S(\vec{x},p,\mu,t)=p^{-2}\frac{\partial}{\partial x_\alpha}p^2 D_{\alpha\sigma}\frac{\partial f_0}{\partial x_\sigma} \tag{48}$$

where, for ease of notation, we have dropped the $t^*$-notation, keeping in mind that the position and time variables are to be taken in the lab coordinate system. Recall that $x_\alpha, x_\sigma \in [p, \mu, X, Y]$, so that 16 different Fokker-Planck coefficients remain.

## 3.9. Interlude

Eq. (43) in a medium at rest and Eq. (48) in a moving medium are general Fokker-Planck transport equations for the Larmor-phase-averaged particle phase space density, which were based on six physical assumptions about the fluctuating electromagnetic field turbulence:

- (1) adiabatic approximation,
- (2) homogeneous and quasi-stationary turbulence,
- (3) existence of a small enough finite decorrelation time of second-order correlation functions,
- (4) random phase (between particles and fluctuations),
- (5) strongly magnetized systems with $B_0 \gg \delta B$
- (6) parallel flows with respect to $\vec{B}_0$.

For all distant cosmic objects (outside the solar system) it is impossible to check on the validity of these assumptions.

Further generalizations including nonparallel flows, shear flows, partially random flows are worth investigating.

Solving the Fokker-Planck equations analytically is mathematically equivalent to the numerical solution of the corresponding system of Ito′s stochastic differential equations (Gardiner 1983).

# 4. Fokker-Planck coefficients

The $\phi$-integrations in the gyro-averaged Fokker-Planck coefficients (42) can be expressed in terms of Bessel functions of the first kind (see RS 2010) using the identity

$$e^{\imath z \sin\eta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{\imath n\eta} \tag{49}$$

and the addition theorems

$$J_0(\lambda R) = \sum_{m=-\infty}^{\infty} J_m(\lambda r_1) J_m(\lambda r_2) e^{\imath m\theta}, \quad R = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta},$$

$$J_\nu(\lambda R) e^{\imath\nu\gamma} = \sum_{n=-\infty}^{\infty} J_n(\lambda r_1) J_{n+\nu}(\lambda r_2) e^{\imath n\theta}, \quad \sin\gamma = \frac{r_1}{R}\sin\theta \tag{50}$$

For magnetostatic turbulence ($\delta E_i = 0$) and axisymmetric turbulence $C_{\alpha,\sigma}(\vec{k},\tau) = P_{\alpha,\sigma}(k_\parallel, k_\perp, \tau)$ one obtains

$$D_{\mu\mu} = \frac{\pi\Omega^2(1-\mu^2)}{B_0^2}\Re\int_0^\infty d\tau\int_{-\infty}^\infty dk_\parallel\int_0^\infty dk_\perp\, k_\perp J_0\left(\frac{k_\perp v_\perp}{\Omega}\left[2(1-\cos\Omega\tau)\right]^{1/2}\right)$$
$$\times\left[e^{\imath(v\mu k_\parallel+\Omega)\tau}P_{LL}(k_\parallel,k_\perp,\tau)+e^{\imath(v\mu k_\parallel-\Omega)\tau}P_{RR}(k_\parallel,k_\perp,\tau)\right]$$
$$= \frac{\pi\Omega^2(1-\mu^2)}{B_0^2}\Re\sum_{n=-\infty}^{\infty}\int_{-\infty}^\infty dk_\parallel\int_0^\infty dk_\perp\, k_\perp\int_0^\infty d\tau e^{-\imath(n\Omega+k_\parallel v_\parallel)\tau}$$
$$\times\left[J_{n-1}^2\left(\frac{k_\perp v_\perp}{\Omega}\right)P_{LL}(k_\parallel,k_\perp,\tau)+J_{n+1}^2\left(\frac{k_\perp v_\perp}{\Omega}\right)P_{RR}(k_\parallel,k_\perp,\tau)\right] \tag{51}$$

First form ideal for numerical computations avoiding infinite sums and calculating a strict upper limit using $J_0(A)\le 1$ and $\cos(A)\le 1$ for all arguments $A$.

With the positively counted damping rate

$$P_{LL,RR}(k_\parallel,k_\perp,\tau) = P^0_{LL,RR}(k_\parallel,k_\perp)e^{(\imath\omega_R(k_\parallel,k_\perp)-\Gamma_{LH,RH}(k_\parallel,k_\perp))\tau}, \tag{52}$$

the $\tau$-integration in the second form of Eq. (51) provides the resonance function

$$\mathcal{R}(\omega_R,\Gamma_{LH,RH}) = \Re\int_0^\infty d\tau e^{-[\imath(n\Omega+v\mu k_\parallel-\omega_R)+\Gamma_{LH,RH}]\tau}$$
$$= \frac{\Gamma_{LH,RH}}{(v\mu k_\parallel-\omega_R+n\Omega)^2+\Gamma^2_{LH,RH}}, \tag{53}$$

so that in the weak-damping limit

$$\lim_{\Gamma_{LH,RH}\to 0} \mathcal{R}(\omega_R, \Gamma_{LH,RH}) = \pi\delta(v\mu k_\parallel - \omega_R + n\Omega) \tag{54}$$

Hence, in this limit

$$D_{\mu\mu} = \frac{\pi^2\Omega^2(1-\mu^2)}{B_0^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_\parallel \int_0^\infty dk_\perp\, k_\perp$$
$$\left[J_{n-1}^2\left(\frac{k_\perp v_\perp}{\Omega}\right) P_{LL}^0(k_\parallel, k_\perp)\delta(v\mu k_\parallel - \omega_{R,LH} + n\Omega)\right.$$
$$\left. + J_{n+1}^2\left(\frac{k_\perp v_\perp}{\Omega}\right) P_{RR}^0(k_\parallel, k_\perp)\delta(v\mu k_\parallel - \omega_{R,RH} + n\Omega)\right] \tag{55}$$

Further evaluation requires the specification of the turbulence geometry (distribution in $k_\parallel$ and $k_\perp$), entering $P_{LL}^0$ and $P_{RR}^0$, and of the dispersion relations of LH and RH polarized collective wave modes. In RS (2002) this and other Fokker-Planck coefficients are calculated for slab ($k_\perp = 0$) waves and isotropically distributed waves.

For slab waves one uses $J_n(z) = \delta_{n,0}$, so that

$$D_{\mu\mu}^{\mathrm{slab}} = \frac{\pi^2\Omega^2(1-\mu^2)}{B_0^2} \int_{-\infty}^{\infty} dk_\parallel \int_0^\infty dk_\perp\, k_\perp$$
$$\left[P_{LL}^0(k_\parallel)\delta(v\mu k_\parallel - \omega_{R,LH} + \Omega) + P_{RR}^0(k_\parallel)\delta(v\mu k_\parallel - \omega_{R,RH} - \Omega)\right] \tag{56}$$

One needs to account for the charge sign of the CR particle (entering through the relativistic gyrofrequency) and the intensities of forward and backward moving, RH- and LH-waves (see e.g. RS 1989, Dung and RS 1990). Figs. 1 and 2 show two illustrative examples.

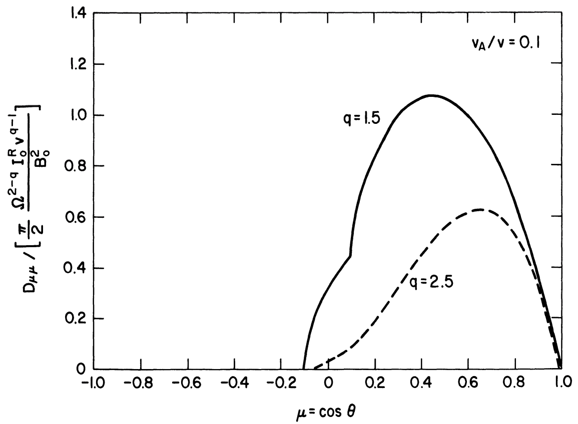


Figure 1: Pitch angle scattering coefficient for CR protons due to RH polarized Alfven waves streaming with equal intensities in both directions. From RS (1989).

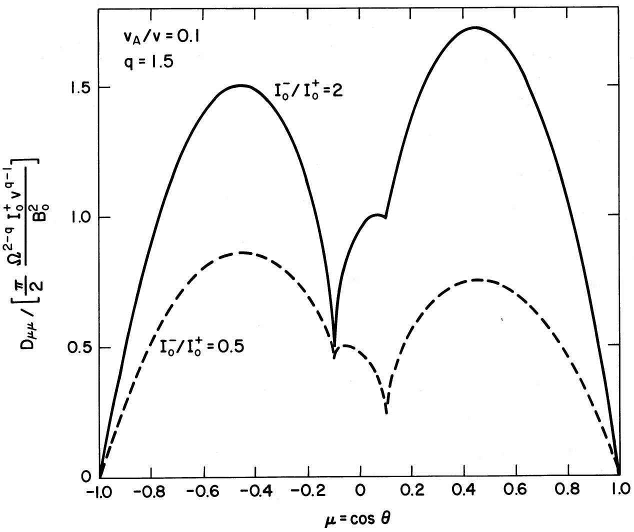


Figure 2: Pitch angle scattering coefficient for CR protons due to RH and LH polarized Alfven waves streaming with different intensities in both directions. From RS (1989).

Both RH- and LH-polarized slab Alfven waves are needed to scatter CR protons (and CR electrons with Lorentzfactor $\gamma > 1836$) at all pitch angles $\mu$.

## 5. Diffusion approximation

Our earlier qualitative estimate of Fokker-Planck coefficients for energetic particles with $v \gg V_A$ indicated that the pitch angle Fokker-Planck coefficient $D_{\mu\mu}$ is the largest one.

We therefore make the basic assumption of diffusion theory that the gyrotropic particle distribution function $f_0(\vec{X}, p, \mu, t)$ under the action of low-frequency magnetohydrodynamic waves adjusts very quickly to a distribution function through pitch-angle diffusion which is close to the isotropic distribution in the rest frame of the moving background plasma. Defining the isotropic part of the phase space density $F(\vec{X}, z, p, t)$ as the $\mu$-averaged phase space density

$$F(\vec{X}, p, t) \equiv \frac{1}{2} \int_{-1}^{1} d\mu \; f_0(\vec{X}, p, \mu, t), \tag{57}$$

we follow the analysis of Jokipii (1966) and Hasselmann and Wibberenz (1968) to split the total density $f_0$ into the isotropic part $F$ and an anisotropic part $g$,

$$f_0(\vec{X}, p, \mu, t) = F(\vec{X}, p, t) + \; g(\vec{X}, p, \mu, t), \tag{58}$$

where because of Eq. (57)

$$\int_{-1}^{1} d\mu \, g(\vec{X}, p, \mu, t) = 0 \tag{59}$$

Singleing out the phase space variable $\mu$ by introducing the reduced set of variables $z_{\alpha,\sigma} \in [X, Y, p]$ the Larmor-phase averaged Fokker-Planck coefficients transport equation (48) reads

$$\Gamma\left[1+\frac{Uv\mu}{c^2}\right]\left[\frac{\partial f_0}{\partial t}-\frac{1}{v}\frac{\partial U}{\partial t}\Gamma^2\left(\mu p\frac{\partial f_0}{\partial p}+(1-\mu^2)\frac{\partial f_0}{\partial \mu}\right)\right]$$
$$+\Gamma\left[U+v\mu\right]\left(\frac{\partial f_0}{\partial Z}-\frac{1}{v}\frac{\partial U}{\partial z}\Gamma^2\left(\mu p\frac{\partial f_0}{\partial p}+(1-\mu^2)\frac{\partial f_0}{\partial \mu}\right)\right]$$
$$+\mathcal{N}f_0+\mathcal{R}f_0-S(\vec{X},p,\mu,t)=$$
$$p^{-2}\frac{\partial}{\partial z_\alpha}p^2D_{\alpha\sigma}\frac{\partial f_0}{\partial z_\sigma}+\frac{\partial}{\partial \mu}D_{\mu\mu}\frac{\partial f_0}{\partial \mu}+\frac{\partial}{\partial \mu}D_{\mu\sigma}\frac{\partial f_0}{\partial z_\sigma}+p^{-2}\frac{\partial}{\partial z_\alpha}p^2D_{\alpha\mu}\frac{\partial f_0}{\partial \mu} \quad (60)$$

Instead of manipulating this complicated equation, we follow the historical development and consider simplified versions of the full transport equation.

**Upshot**: the more terms we keep in the Fokker-Planck transport equation the more terms result in the diffusion-convection transport equation for the isotropic part $F(\vec{X}, p, t)$ result.

## 5.1. Magnetostatic turbulence in a medium at rest (Jokipii 1966)

If only magnetostatic turbulence ($\delta\vec{E}=0$) is considered, $D_{\mu\mu}$ is the only non-vanishing Fokker-Planck coefficient. Ignoring the effects of radiation losses, the mirror force, adopting an isotropic source term, and considering a medium at rest, the Fokker-Planck equation (60) simplifies to

$$\frac{\partial f_0}{\partial t}+v\mu\frac{\partial f_0}{\partial Z}-S(\vec{X},p,t)=\frac{\partial}{\partial\mu}D_{\mu\mu}\frac{\partial f_0}{\partial\mu} \tag{61}$$

Inserting the ansatz (58) provides

$$\frac{\partial F}{\partial t}+v\mu\frac{\partial F}{\partial Z}+\frac{\partial g}{\partial t}+v\mu\frac{\partial g}{\partial Z}-S(\vec{X},p,t)=\frac{\partial}{\partial\mu}D_{\mu\mu}\frac{\partial g}{\partial\mu} \tag{62}$$

Now averaging over $\mu$ yields (note that $D_{\mu\mu}\propto(1-\mu^2)$ becomes zero for $\mu\to\pm1$)

$$\frac{\partial F}{\partial t}+\frac{v}{2}\frac{\partial}{\partial Z}\int_{-1}^{1}d\mu\mu g-S(\vec{X},p,t)=0 \tag{63}$$

Next we subtract Eq. (63) from Eq. (62) to obtain

$$v\mu[\frac{\partial F}{\partial Z}+\frac{\partial g}{\partial Z}]+\frac{\partial g}{\partial t}-\frac{v}{2}\frac{\partial}{\partial Z}\int_{-1}^{1}d\mu\mu g=\frac{\partial}{\partial\mu}D_{\mu\mu}\frac{\partial g}{\partial\mu}, \tag{64}$$

which together with Eq. (63) is still exact.

### 5.1.1. Anisotropy

The diffusion approximation applies if the isotropic particle density is slowly evolving, i.e. $(\partial F/\partial t) = \mathcal{O}(F/T)$ and $(\partial F/\partial Z) = \mathcal{O}(F/L_0)$ with typical length scales $L_0 >> \lambda$ and time scales $T >> \tau$ much larger than the mean free path $\lambda = v\tau$ and the pitch angle scattering relaxation time $\tau \simeq \mathcal{O}(1/D_{\mu\mu})$, respectively. In this case the particles have enough time to adjust locally to a near-equilibrium, so that the anisotropy is small i.e. $g << F$.

If we then regard $g$ as of order $\tau$, when $F$ is of order $1$, we may characterize the differential operators in Eq. (64) by different time scales. Therefore to lowest order we approximate Eq. (64) by

$$v\mu\frac{\partial F}{\partial Z} \simeq \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial g}{\partial \mu}\right] \tag{65}$$

Integrating over $\mu$ we obtain

$$D_{\mu\mu}\frac{\partial g}{\partial \mu} = c_1 + \frac{v\mu^2}{2}\frac{\partial F}{\partial Z}, \tag{66}$$

where the integration constant $c_1 = -v/2(\partial F/\partial Z)$ is determined from the requirement that the left-hand side of Eq. (66) vanishes for $\mu = \pm 1$, yielding

$$\frac{\partial g}{\partial \mu} = -\frac{v}{2}\frac{1-\mu^2}{D_{\mu\mu}(\mu)}\frac{\partial F}{\partial Z} \tag{67}$$

A further integration provides

$$g(\vec{X}, p, \mu, t) \simeq c_2 - \frac{v}{2}\frac{\partial F}{\partial Z}\int_{-1}^{\mu} ds\, \frac{1-s^2}{D_{\mu\mu}(s)}, \tag{68}$$

where the condition (59), i.e. $\int_{-1}^{1} d\mu g(\mu) = 0$, determines $c_2$. The CR anisotropy (68) then becomes

$$g(\vec{X}, p, \mu, t) = \frac{v}{4}\frac{\partial F}{\partial Z}\left[\int_{-1}^{1} d\mu \frac{1-\mu^2}{D_{\mu\mu}(\mu)} - 2\int_{-1}^{\mu} ds\, \frac{1-s^2}{D_{\mu\mu}(s)}\right], \tag{69}$$

resulting from the gradient of $F$ with respect to $Z$ ("streaming anisotropy").

### 5.1.2. Diffusion transport equation

The anisotropy (69) readily provides the first moment

$$\begin{aligned}\int_{-1}^{1} d\mu\, \mu g(\mu) &= -\frac{v}{2}\frac{\partial F}{\partial Z}\int_{-1}^{1} d\mu\, \mu \int_{-1}^{\mu} ds\, \frac{1-s^2}{D_{\mu\mu}(s)} \\ &= -\frac{v}{2}\frac{\partial F}{\partial Z}\int_{-1}^{1} d\mu\, \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)}, \end{aligned} \tag{70}$$

where we partially integrated with respect to $\mu$. Inserting the result in Eq. (63)

provides the diffusion transport equation for the isotropic part of the phase space distribution

$$\frac{\partial F}{\partial t} - \frac{\partial}{\partial Z}[\kappa_\| \frac{\partial F}{\partial Z}] = S(\vec{X}, p, t), \tag{71}$$

with the parallel spatial diffusion coefficient

$$\kappa_\| = \frac{v\lambda_\|}{3} = \frac{v^2}{8} \int_{-1}^{1} d\mu \, \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)} \tag{72}$$

determined by the pitch-angle average of the Fokker-Planck coefficient $D_{\mu\mu}$.

## 5.2. Magnetostatic turbulence in a medium at rest with weak adiabatic focusing (Earl 1976)

Now we include the adiabatic focusing term from the mirror force, so that the Fokker-Planck equation (60) reduces to

$$\frac{\partial f_0}{\partial t} + v\mu \frac{\partial f_0}{\partial Z} + \frac{v(1-\mu^2)}{2L_3} \frac{\partial f_0}{\partial \mu} - S(\vec{X}, p, t) = \frac{\partial}{\partial \mu} D_{\mu\mu} \frac{\partial f_0}{\partial \mu} \tag{73}$$

Repeating the analysis of the last subsection provides

$$\frac{\partial F}{\partial t} + \frac{v}{2}[\frac{\partial}{\partial Z} + \frac{1}{L_3}] \int_{-1}^{1} d\mu \mu g - S(\vec{X}, p, t) = 0 \tag{74}$$

and

$$v\mu[\frac{\partial F}{\partial Z}+\frac{\partial g}{\partial Z}]+\frac{\partial g}{\partial t}+\frac{v\mu g}{L_3}-\frac{v}{2}[\frac{\partial}{\partial Z}+\frac{1}{L_3}]\int_{-1}^{1} d\mu\mu g=\frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial g}{\partial \mu}-\frac{v(1-\mu^2)g}{2L_3}\right), \tag{75}$$

which we approximate in the limit of weak focusing again by the streaming anisotropy (69). With the same first moment (70) we find the pseudo-diffusion equation for the isotropic part of the phase space distribution

$$\frac{\partial F}{\partial t}-\frac{\partial}{\partial Z}[\kappa_{\|}\frac{\partial F}{\partial Z}]-\frac{\kappa_{\|}}{L_3}\frac{\partial F}{\partial Z}=S(\vec{X},p,t), \tag{76}$$

where the spatial convection speed $V=\kappa_{\|}/L_3$ is positive in a diverging guide magnetic field and negative in a converging guide magnetic field.

## 5.3. Magnetostatic turbulence in a moving medium with $U \ll c$ with weak adiabatic focusing (Litvinenko and RS 2013)

For magnetostatic turbulence the Fokker-Planck equation (60) in a moving medium with nonrelativistic speed $U \ll c$ reduces to

$$\frac{\partial f_0}{\partial t}+[U+v\mu]\left[\frac{\partial f_0}{\partial Z}-\frac{1}{v}\frac{\partial U}{\partial Z}\left(\mu p\frac{\partial f_0}{\partial p}+(1-\mu^2)\frac{\partial f_0}{\partial \mu}\right)\right]$$

$$+\frac{v\mu f_0}{L_3}=\frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f_0}{\partial \mu}-\frac{v(1-\mu^2)f_0}{2L_3}\right)-S(\vec{X},p,t) \tag{77}$$

Substituting the ansatz (58) provides after averaging and subtraction

$$\frac{\partial F}{\partial t} + U\frac{\partial F}{\partial Z} - \frac{1}{3}\frac{\partial U}{\partial Z}p\frac{\partial F}{\partial p} + \frac{v}{2}[\frac{\partial}{\partial Z} + \frac{1}{L_3}]\int_{-1}^{1} d\mu\mu g$$
$$-\frac{1}{2}\frac{\partial U}{\partial Z}\left[\frac{U}{v}(2+p\frac{\partial}{\partial p})\int_{-1}^{1} d\mu\mu g + (3+p\frac{\partial}{\partial p})\int_{-1}^{1} d\mu\mu^2 g\right] = S(\vec{X},p,t) \quad (78)$$

In the weak focusing limit we use the streaming anisotropy (69), implying besides the first moment (70)

$$\int_{-1}^{1} d\mu\mu^2 g = -\frac{v}{6}\frac{\partial F}{\partial Z}\int_{-1}^{1} d\mu\frac{\mu(1-\mu^2)^2}{D_{\mu\mu}(\mu)} \quad (79)$$

Consequently, Eq. (78) becomes

$$\frac{\partial F}{\partial t} + [U - \frac{\kappa_\parallel}{L_3}]\frac{\partial F}{\partial Z} - \frac{1}{3}\frac{\partial U}{\partial Z}p\frac{\partial F}{\partial p} - \frac{\partial}{\partial Z}[\kappa_\parallel\frac{\partial F}{\partial Z}] + \frac{U}{v}\frac{\partial U}{\partial Z}(2+p\frac{\partial}{\partial p})[\frac{\kappa_\parallel}{v}\frac{\partial F}{\partial Z}]$$
$$+\frac{1}{12}\frac{\partial U}{\partial Z}(3+p\frac{\partial}{\partial p})[v\int_{-1}^{1} d\mu\frac{\mu(1-\mu^2)^2}{D_{\mu\mu}(\mu)}\frac{\partial F}{\partial Z}] = S(\vec{X},p,t) \quad (80)$$

For $U \ll v$ and symmetric Fokker-Planck coefficients $D_{\mu\mu}(-\mu) = D_{\mu\mu}(\mu)$ the last two terms on the LHS can be ignored and we obtain the **Parker diffusion-convection equation**

$$\frac{\partial F}{\partial t}+\left[U-\frac{\kappa_\parallel}{L_3}\right]\frac{\partial F}{\partial Z}-\frac{1}{3}\frac{\partial U}{\partial Z}p\frac{\partial F}{\partial p}-\frac{\partial}{\partial Z}[\kappa_\parallel\frac{\partial F}{\partial Z}]=S(\vec{X},p,t) \tag{81}$$

Note that the momentum convection term leads to cooling of particles in expanding flows with positive $\partial U/\partial z > 0$ as in the solar wind, but to particle acceleration in converging flows with negative $\partial U/\partial z < 0$, which is the physical reason for diffusive shock acceleration (first-order Fermi acceleration).

## 5.4. Full transport equation for nonrelativistic flows (RS and Shalchi 2008, RS 1989, Skilling 1975)

We now give up on the magnetostatic approximation and include finite electric field effects. The diffusion approximation of the Fokker-Planck equation (60) leads to

$$\frac{\partial F}{\partial t}+\mathcal{R}F-S(\vec{X},p,t)-\begin{pmatrix}V_X\\V_Y\\V_Z\\V_p\end{pmatrix}\cdot\begin{pmatrix}\partial_X F\\\partial_Y F\\\partial_z F\\p^{-2}\partial_p p^2 F\end{pmatrix}$$

$$=\begin{pmatrix}\partial_X\\\partial_Y\\\partial_z\\p^{-2}\partial_p p^2\end{pmatrix}\cdot\begin{pmatrix}\kappa_{XX}&\kappa_{XY}&\kappa_{XZ}&\kappa_{Xp}\\\kappa_{YX}&\kappa_{YY}&\kappa_{YZ}&\kappa_{Yp}\\\kappa_{ZX}&\kappa_{ZY}&\kappa_{ZZ}&\kappa_{Zp}\\\kappa_{pX}&\kappa_{pY}&\kappa_{pZ}&\kappa_{pp}\end{pmatrix}\begin{pmatrix}\partial_X F\\\partial_Y F\\\partial_z F\\p^{-2}\partial_p p^2 F\end{pmatrix} \tag{82}$$

We identify the individual pitch-angle averaged convection speeds

$$V_X = \frac{\kappa_{ZX}}{L_3} + \frac{\gamma+1}{\gamma v^2} U \frac{\partial U}{\partial z} \kappa_{ZX}, \; V_Y = \frac{\kappa_{ZY}}{L_3} + \frac{\gamma+1}{\gamma v^2} U \frac{\partial U}{\partial z} \kappa_{ZY},$$

$$V_Z = -U + \frac{\kappa_{ZZ}}{L_3} + \frac{\gamma+1}{2\gamma v^2} U \frac{\partial U}{\partial z} \kappa_{zz}, \; V_p = \frac{1}{3}\frac{\partial U}{\partial z} p + \frac{\kappa_{Zp}}{L_3} + \frac{\gamma+1}{\gamma v^2} U \frac{\partial U}{\partial z} \kappa_{Zp} \quad (83)$$

and list some of the 16 pitch-angle averaged diffusion coefficients

$$\kappa_{ZZ} = \frac{v^2}{8}\int_{-1}^{1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}}, \; \kappa_{Zp} = \frac{v}{4}\int_{-1}^{1} d\mu \frac{(1-\mu^2)D_{\mu p}}{D_{\mu\mu}},$$

$$\kappa_{pp} = \frac{1}{2}\int_{-1}^{1} d\mu \left[D_{pp} - \frac{D_{\mu p}D_{p\mu}}{D_{\mu\mu}}\right]$$

$$-\frac{p}{4}\frac{\partial U}{\partial z}\left[\frac{U}{v}\int_{-1}^{1} d\mu \frac{(1-\mu^2)D_{\mu p}}{D_{\mu\mu}} + \frac{2}{3}\int_{-1}^{1} d\mu \frac{(1-\mu^3)D_{\mu p}}{D_{\mu\mu}}\right],$$

$$\kappa_{XX} = \frac{1}{2}\int_{-1}^{1} d\mu \left[D_{XX} - \frac{D_{X\mu}D_{\mu X}}{D_{\mu\mu}}\right]$$

$$+\frac{\epsilon_a v R_L}{12 L_2}\left[\int_{-1}^{1} d\mu \frac{\mu(1-\mu^2)D_{X\mu}}{D_{\mu\mu}} - \int_{-1}^{1} d\mu \frac{(1-\mu^3)D_{\mu X}}{D_{\mu\mu}}\right]$$

$$-\frac{v^2 R_L^2}{72 L_2^2}\left[\int_{-1}^{1} d\mu \frac{\mu(1-\mu^2)(1-\mu^3)}{D_{\mu\mu}}\right.$$

$$\left. + \frac{U}{v}\int_{-1}^{1} d\mu \frac{\mu(1-\mu^2)^2}{D_{\mu\mu}} + \frac{2}{3}\int_{-1}^{1} d\mu \frac{\mu(1-\mu^2)(1-\mu^3)}{D_{\mu\mu}}\right] \quad (84)$$

In its general form the diffusion–convection transport equation (82) contains spatial diffusion and spatial convection terms as well as momentum diffusion and momentum convection terms. Since the pioneering work of Fermi (1949, 1954) it has become customary to refer to the latter two as Fermi acceleration of second and first order, respectively.

With its 16 different diffusion coefficients and 4 convection speeds the general diffusion-convection transport eequation (82) is rather complicated and involved. One has to emphasize that, depending on the type of turbulent electromagnetic fields considered, not all of these 20 CR transport parameters have nonzero values, and some of the transport parameters have much higher values than others, so that simplified versions of the general transport equation (82) are justified.

The first term in the momentum convection term $V_p$ leads to acceleration for converging bulk flow (i.e., $dU/dz < 0$) but to deceleration for expanding flows (i.e., $dU/dz > 0$).

The second term in the momentum convection term $V_p$ leads to focused acceleration of CR particles (RS and Shalchi 2008, Litvinenko 2011) provided the product of $\kappa_{Zp} L_3 < 0$ is negative.

## 6. Focused acceleration

For $L_1 = L_2 = \infty$ and $\kappa_{XX} = \kappa_{YY} = \kappa_{XY} = 0$ the modified diffusion-convection equation (82) in a medium at rest reduces to the focused diffusion-convection transport equation ($L_3 = L$)

$$\frac{\partial F}{\partial t} + \frac{F}{T_c} - S(\vec{X}, p, t) + \frac{\partial}{\partial z}\left(\frac{\kappa_{zz}}{L} F\right) - \frac{1}{p^2}\frac{\partial}{\partial p}\left(\left[p^2 \dot{p}_{\mathrm{loss}} + \frac{a_{zp} p^2}{L}\right] F\right)$$
$$= \begin{pmatrix} \partial_X \\ \partial_Y \\ \partial_z \\ p^{-2}\partial_p p^2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_{zz} & a_{zp} \\ 0 & 0 & -a_{zp} & A \end{pmatrix} \begin{pmatrix} \partial_X F \\ \partial_Y F \\ \partial_z F \\ \partial_p F \end{pmatrix} \tag{85}$$

with

$$a_{zp} = \kappa_{Zp} = \frac{v}{4}\int_{-1}^{1} d\mu \frac{(1-\mu^2) D_{\mu p}(\mu)}{D_{\mu\mu}(\mu)} = \frac{V_A H}{3} p, \tag{86}$$

for isospectral undamped slab Alfven wave turbulence, where $H = (I^+ - I^-)/(I^+ + I^-)$ denotes the net cross helicity of Alfven waves.

### 6.1. New transport terms due to weak adiabatic focusing

Weak adiabatic focusing gives rise to two terms in Eq. (85) that represent convective transport terms parallel to the guide field and in momentum space, respectively.

The convective term along the guide field has been derived before by Earl (1976) and Kunstmann (1979); the momentum convection term by RS and Shalchi (2008).

- For weak focusing ($|L| \gg \lambda$) the new parallel convective speed $\kappa_{zz}/L = v\lambda_{\parallel}/3L$ is much less than the individual cosmic ray speed $v$.
- Particularly interesting is the new convection term in momentum space

$$\frac{1}{p^2}\frac{\partial}{\partial p}\left[\frac{a_{zp}Lp^2}{L^2}F\right] = \frac{1}{p^2}\frac{\partial}{\partial p}\left[\frac{V_A H}{3L}p^3 F\right]$$

  For positive values of the product $a_{zp}L \propto HL > 0$ it represents a continuous momentum loss term, whereas for negative values $a_{zp}L \propto HL < 0$ it represents a first-order Fermi-type acceleration term. The focusing length $L(z) = -(B_0/(\partial B_0/\partial z))$ is positive for a diverging guide magnetic field and negative for a converging guide field.

- This novel distributed focused acceleration process, which is a 1st order Fermi acceleration process, operates in all cosmic sources with $HL < 0$, including the upstream medium of shock waves, haloes of spiral galaxies and solar flare loops. If it is the dominant transport term it generates the power-law distribution function $F \propto p^{-3}$ in steady-state conditions.
- For $HL > 0$ it represents a deceleration (momentum loss) process. It could prevent (or reduce) diffusive shock acceleration.

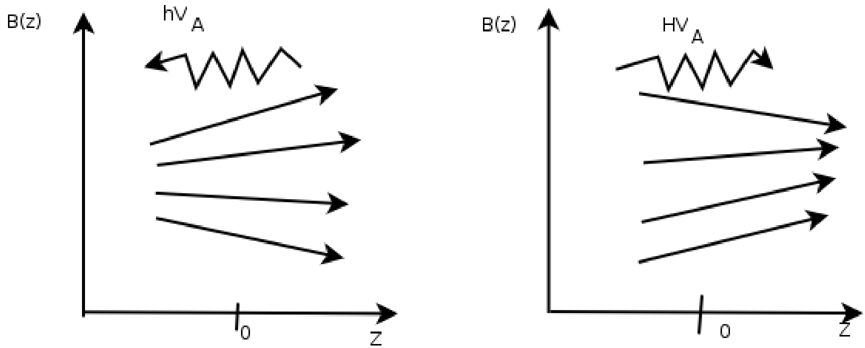


Figure 3: Conditions for 1st-order distributed Fermi acceleration in diverging (a) and converging (b) guide magnetic fields. In diverging magnetic fields a net negative ($h = -H < 0$) cross helicity state of Alfven waves (pronged curve) convects the average particle to regions of stronger field strength. In converging magnetic fields a net positive ($H > 0$) cross helicity state of Alfven waves also convects the average particle to regions of stronger field stength. In both cases the conservation of the pitch-angle averaged magnetic moment of the particle requires the increase of the particle momentum.

# 7. Momentum spectra

In the Parker equation (81) with $L_3 = \infty$ we combine

$$U\frac{\partial F}{\partial z} - \frac{1}{3}\frac{\partial U}{\partial z}p\frac{\partial F}{\partial p} = \frac{\partial}{\partial z}(UF) - \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(\frac{1}{3}\frac{\partial U}{\partial z}p\right)F\right] \tag{87}$$

Allowing for momentum diffusion, momentum losses and a mono-momentum source term then provides in the steady-state case

$$\frac{\partial}{\partial z}\left[\kappa_\| \frac{\partial F}{\partial z} - UF\right] - \frac{F}{T_c} + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\kappa_{pp}\frac{\partial F}{\partial p} + p^2\left(\dot{p}_{\rm gain} - \dot{p}_{\rm loss}\right)F\right] = -S_0(z)\delta(p-p_0) \tag{88}$$

with the 1st-order Fermi acceleration term $\dot{p}_{\rm gain} = a_1 p$ from negative $\frac{\partial U}{\partial z} = -3a_1$ and/or focused acceleration.

For $\kappa_\| = \kappa_0$ independent of $p$ we can expand the solutions in terms of the spatial eigenfunctions (plus finite spatial boundary conditions)

$$F(z,p) = \sum_{n=0}^{\infty} g_n(p)Z_n(z), \quad S_0(z) = \sum_{n=0}^{\infty} a_n Z_n(z) \tag{89}$$

where

$$\frac{\partial}{\partial z}\left[\kappa_0\frac{\partial Z_n}{\partial z} - UZ_n\right] + \lambda_n Z_n(z) = 0. \tag{90}$$

Each of the momentum dependent expansion coefficients obeys the **leaky-box** equation

$$\frac{1}{p^2}\frac{d}{dp}\left[p^2\kappa_{pp}\frac{dg_n}{\partial p}+p^2\left(\dot{p}_{\rm gain}-\dot{p}_{\rm loss}\right)g_n\right]-(\lambda_n+T_c^{-1})g_n=-a_n\delta(p-p_0) \quad (91)$$

## 7.1. Only 1st-order Fermi acceleration

For $\dot{p}_{\rm loss}=0$, $\kappa_{pp}=0$, 1st-order Fermi acceleration generates the power-law solution for $p\geq p_0$

$$g_n(p)\propto p^{-s_n}, \quad s_n=3+\frac{\lambda_n+T_c^{-1}}{a_1}, \quad (92)$$

where $s_n\to 3$ for efficient acceleration $a_1\to\infty$. Note that the smallest eigenvalue $\lambda_0$ dominates the momentum dependence of $F(z,p)$ at large momenta $p\gg p_0$.

## 7.2. Only 2nd-order Fermi acceleration

For $\dot{p}_{\rm loss}=0$, $a_1=0$, and $\kappa_{pp}=\kappa_1 p^{2-\eta}$ and $\lambda_n+T_c^{-1}=\lambda_0 p^b$, we obtain the Bessel function solution (Pikelner and Tsytovich 1976, Barbosa 1979)

$$g_n(p\geq p_0)\propto p^{\frac{\eta-3}{2}}K_{\left|\frac{3-\eta}{b+\eta}\right|}\left(\frac{2}{|b+\eta|}\sqrt{\frac{\lambda_0}{\kappa_1}}p^{\frac{b+\eta}{2}}\right), \quad (93)$$

which is a power law $\propto p^{\eta-3}$ at small arguments with an exponential cut-off. For more general solutions of Eq. (91) see RS (1984, 2002).

## 7.3. Negligible spatial convection

If at large enough CR momenta $UF \ll \kappa_2 p^\eta \frac{\partial F}{\partial z}$ , Eq. (88) reduces to

$$\kappa_2 \frac{\partial^2 F}{\partial z^2} - \frac{F}{T_c p^\eta} + \frac{1}{p^{2+\eta}} \frac{\partial}{\partial p} \left[ p^2 \kappa_{pp} \frac{\partial F}{\partial p} + p^2 \left( \dot{p}_{\text{gain}} - \dot{p}_{\text{loss}} \right) F \right] = - \frac{S_0(z)\delta(p-p_0)}{p_0^\eta}, \tag{94}$$

which again can be expanded as in Eq. (89)

$$F(z,p) = \sum_{n=0}^{\infty} G_n(p) U_n(z), \quad S_0(z) = \sum_{n=0}^{\infty} w_n U_n(z) \tag{95}$$

with

$$\kappa_2 \frac{\partial^2 U_n(z)}{\partial z^2} + \lambda_n U_n(z) = 0. \tag{96}$$

In this case the leaky-box equations read

$$\frac{1}{p^2} \frac{d}{dp} \left[ p^2 \kappa_{pp} \frac{dG_n}{\partial p} + p^2 \left( \dot{p}_{\text{gain}} - \dot{p}_{\text{loss}} \right) G_n \right] - (\lambda_n p^\eta + T_c^{-1}) G_n = -w_n \delta(p - p_0) \tag{97}$$

# 8. Summary and conclusions

- We have reviewed the fundamentals of cosmic ray astrophysics stressing the importance of electromagnetic acceleration and transport processes with $B_0 \gg \delta B \gg \delta E$.

- The ordering $B_0 \gg \delta B \gg \delta E$, necessary for explaining the observed nearly isotropic CR momentum distribution function, is the basis for a perturbation scheme leading to the modified diffusion-convection CR transport equation that describes all electromagnetic acceleration and transport processes discussed today.

- The actual determination of the Fokker-Planck coefficients requires the knowledge of the second-order electromagnetic correlation functions, either from observations in the interplanetary medium or from fluctuation theory.

- The theory of CR acceleration and transport is an active field of research with many contributions still to be made until a full understanding of the origin of cosmic rays is achieved. It is a pity that these days far too little young scientists work on analytical kinetic theory.